

A note on Chow stability of the Projectivisation of Gieseker Stable Bundles

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Abstract

We investigate Chow stability of projective bundles $\mathbb{P}(E)$ where E is a strictly Gieseker stable bundle over a base manifold that has constant scalar curvature. We show that, for suitable polarisations \mathcal{L} , the pair $(\mathbb{P}(E), \mathcal{L})$ is Chow stable and give examples for which it is not asymptotically Chow stable.

1 Introduction

Our aim is to investigate the connection between stability of a vector bundle E and stability of the projective bundle $\mathbb{P}(E)$ as a polarised manifold. Roughly speaking one expects that $\mathbb{P}(E)$ is stable, with respect to polarisations that make the fibres sufficiently small, if and only if E is a stable vector bundle over a base that is stable as a manifold.

The first result along these lines is due to I. Morrison [21] who showed that if E is a stable rank 2 bundle on a smooth Riemann surface B then the ruled surface $\pi: \mathbb{P}(E) \rightarrow B$ is Chow stable with respect to the polarisation $L_k = \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* \mathcal{O}_B(k)$ for $k \gg 0$. Later, building on the work of E. Calabi, A. Fujiki, C. Lebrun and many others, V. Apostolov, D. Calderbank, P. Gauduchon and C. Tønnesen-Friedman have provided a complete understanding of the situation for higher rank bundles over a smooth Riemann surface. They show there is a constant scalar curvature Kähler metric (cscK in short) in any Kähler class on $\mathbb{P}(E)$ if and only if the bundle E is Mumford polystable [1, 2, 3]. Such metrics are related to stability through the Yau-Tian-Donaldson conjecture (see, for example, [26] for an account). In particular it implies through work of S.K. Donaldson [8] that $\mathbb{P}(E)$ is asymptotically Chow stability, by which one means that if r is sufficiently large then the embedding of $\mathbb{P}(E)$ into projective space using the linear series determined by L_k^r is Chow stable (see also A. Della Vedova and F. Zuddas [5] for a generalisation).

There are at least two extensions to the case when the base B has higher dimension. First, a result of Y.-J. Hong [14] states if E is a Mumford-stable

bundle of any rank over a smooth base B that has a discrete automorphism group and a cscK metric, then $\mathbb{P}(E)$ will also admit a cscK metric, again making the fibres small. (Once again, from [8], this implies that $\mathbb{P}(E)$ is asymptotically Chow stable.) Second, a result of R. Seyyedali states that in fact under these conditions, $\mathbb{P}(E)$ is Chow stable with respect to L_k for $k \gg 0$, the novelty here being that stability is not taken asymptotically, which implies Morrison's result.

The purpose of this note is to relax the assumption that E is Mumford stable and instead consider bundles that are merely Gieseker stable. To state the theorems precisely, let B be a smooth polarised manifold carrying an ample line bundle L such that the automorphism group $\text{Aut}(B, L)/\mathbb{C}^*$ is discrete. The projective bundle $\pi: \mathbb{P}(E) \rightarrow B$ carries a tautological bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$, and the line bundle

$$\mathcal{L}_k := \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* L^k$$

is ample for k sufficiently large.

Theorem 1. *Suppose that E is Gieseker stable and its Jordan-Hölder filtration is given by subbundles, and assume there is a constant scalar curvature Kähler metric in the class $c_1(L)$. Then $(\mathbb{P}(E), \mathcal{L}_k)$ is Chow stable for k sufficiently large.*

Theorem 2. *Suppose that E is a rank 2 bundle over a surface and F is a subbundle of E such that E/F is locally free. Suppose furthermore $\mu(F) = \mu(E)$ and*

$$4 (\text{ch}_2(E)/2 - \text{ch}_2(F)) + c_1(B) (c_1(E)/2 - c_1(F)) < 0,$$

where ch_2 denotes the degree 2 term in the Chern character. Then for k sufficiently large, $(\mathbb{P}(E), \mathcal{L}_k)$ is not K -semistable and thus not asymptotic Chow stable.

These theorems should be compared to an observation of D. Mumford that a quartic cuspidal plane curve is Chow stable (as a plane curve), but not asymptotically Chow stable [22, Section 3]. We consider the results here noteworthy insofar as it gives smooth examples of the same nature (see Section 5).

When E is Mumford stable Theorem 1 is due to Seyyedali [28] which in turn builds on work of Donaldson [8]. Our proof will be along the same lines, the main innovation being to replace the Hermitian-Einstein metrics used by Seyyedali with the almost Hermitian-Einstein metrics on a Gieseker stable bundle furnished by N.C. Leung [16]. Under the above assumptions, we construct a sequence of metrics that are balanced i.e make the Bergman function for $(\mathbb{P}(E), \mathcal{L}_k)$ constant. The proof of Theorem 2 consists of a calculation of the Futaki invariant similar to that of Ross-Thomas [26].

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Conventions: If $\pi: E \rightarrow B$ is a vector bundle then $\pi: \mathbb{P}(E) \rightarrow B$ shall denote the space of complex *hyperplanes* in the fibres of E . Thus $\pi_* \mathcal{O}_{\mathbb{P}(E)}(r) = S^r E$ for $r \geq 0$.

2 Preliminaries

Before discussing almost Hermitian Einstein metrics we recall some basic definitions. Let (B, L) be a polarised manifold with $b = \dim B$ and $E \rightarrow B$ a vector bundle. We say that E is *Mumford stable* if for all proper coherent subsheaves $F \subset E$

$$\mu(F) < \mu(E)$$

where the slope $\mu(F) = \mu_L(F) = \deg_L F / \text{rk}(F)$ is the quotient of the degree of F (with respect to L) by its rank $\text{rk}(F)$. We say it is *Mumford semistable* if the same condition holds but with non-strict inequality. Finally E is *Mumford polystable* if it is the direct sum of Mumford stable bundles whose factors all have the same slope.

Any Mumford semi-stable bundle E has a Jordan-Hölder filtration $0 = F_0 \subset F_1 \subset \dots \subset F_n = E$ by torsion free subsheaves such that the quotients F_i/F_{i+1} are Mumford stable with $\mu(E) = \mu(F_i/F_{i+1})$. We say that E has a *Jordan-Hölder filtration given by subbundles* if it is Mumford semistable and all the quotients F_i/F_{i+1} are locally free (see [16, Theorem 3]).

We say that E is *Gieseker stable* if for all proper coherent subsheaves $F \subset E$ one has the following inequality for the normalised Hilbert polynomials

$$\frac{\chi(F \otimes L^k)}{\text{rk}(F)} < \frac{\chi(E \otimes L^k)}{\text{rk}(E)} \quad \text{for } k \gg 0,$$

and *Gieseker semistability*, *Gieseker polystability* is defined analogously. It is known that if E is Gieseker stable then it is simple [15], which means that $\text{Ker}(\bar{\partial}) = \text{Ker}(\partial) = \mathbb{C}\text{Id}_E$ [15, 17].

These stability notions are related; using that $\mu_L(F)$ is the leading order term in k of $\chi(F \otimes L^k)/\text{rk}(F)$ one sees immediately that

$$\begin{array}{ccccccc} \text{Mumford} & \Rightarrow & \text{Gieseker} & \Rightarrow & \text{Gieseker} & \Rightarrow & \text{Mumford} \\ \text{stable} & & \text{stable} & & \text{semistable} & & \text{semistable} \end{array}.$$

2.1 Almost Hermitian-Einstein metrics and Gieseker stability

Now suppose L is equipped with a smooth hermitian metric h_L with curvature $\omega := c_1(h_L) > 0$.

Definition 3. We say that a sequence of hermitian metrics H_k on E is *almost Hermitian-Einstein* if for each $r \geq 0$ the curvature F_{H_k} is bounded in the C^r -norm uniformly with respect to k , and furthermore

$$[e^{F_{H_k} + k\omega \text{Id}_E} \text{Todd}(B)]^{(b,b)} = \frac{\chi(E \otimes L^k)}{\text{rk}(E)} \text{Id}_E \frac{\omega^b}{b!}. \quad (1)$$

In the above the (b, b) indicates taking the top order forms on the left hand side, and $\text{Todd}(B) = 1 + c_1(B) + \frac{1}{2}(c_1(B)^2 + c_2(B)) + \dots$ is the harmonic representative of the Todd class with respect to ω .

By a simple rearrangement this condition implies

$$\sqrt{-1}\Lambda_\omega F_{H_k} - \mu(E)\text{Id}_E = T_0 + k^{-1}T_1 + \dots \quad (2)$$

where $T_i \in C^\infty(\text{End}(E))$ are hermitian quantities depending on F_{H_k} and ω that are bounded uniformly over k in the C^r -norm. Moreover we can arrange so

$$T_0 = -\frac{\text{scal}(\omega)}{2}\text{Id}_E. \quad (3)$$

where $\text{scal}(\omega)$ is the scalar curvature of ω .

Remark 4. By the C^r -norm above we mean the sum of the supremum norms of the first r derivatives taken using the pointwise operator norm with respect to a background metric on the bundle in question (that should be fixed once and for all). From now on we shall write $O_{C^r}(k^i)$ to mean a sum of terms bounded in the C^r -norm by Ck^i for some constant C . Thus, in the above, $T_i = O_{C^r}(k^0) = O_{C^r}(1)$.

In [16], Leung proved a Kobayashi-Hitchin type correspondence for Gieseker stable vector bundles.

Theorem 5 (Leung). *Assume that the Jordan-Hölder filtration of E is given by subbundles. Then E is Gieseker stable if and only if E admits a sequence of almost Hermitian-Einstein metrics for $k \gg 0$.*

For simplicity we package together the following assumption:

(A) Let E be an Gieseker stable holomorphic vector bundle of rank $\text{rk}(E)$ whose Jordan-Hölder filtration is given by subbundles.

From now on we will also assume that E is not Mumford stable, otherwise our results are direct consequences of [28, 31, 32].

2.2 Balanced metrics

Suppose now in addition to our metric h_L on L we also have a smooth Hermitian metric H on E . These induce a hermitian metric $H \otimes h_L^k$ on $E \otimes L^k$ which determines an L^2 -inner product on the space of smooth sections $C^\infty(E \otimes L^k)$ given by

$$\|s\|_{L^2}^2 = \int_B |s|_{H \otimes h_L^k}^2 \frac{\omega^b}{b!}.$$

Associated to this data there is a projection operator $P_k: C^\infty(B, E \otimes L^k) \rightarrow H^0(B, E \otimes L^k)$ onto the space of holomorphic sections for each k . The *Bergman kernel* is defined to be the kernel of this operator which satisfies

$$P_k(f)(x) = \int_B B_k(x, y) f(y) \frac{\omega_y^b}{b!} \quad \text{for all } f \in C^\infty(E \otimes L^k),$$

(see [31, Section 4]).

We wish to consider the Bergman kernel restricted to the diagonal, which by abuse of notation we write as $B_k(x) = B_k(x, x)$. Thus $B_k(x)$ lies in $C^\infty(\text{End}(E))$ which we shall refer to as the *Bergman endomorphism* for $E \otimes L^k$, which of course depends on the data $(H \otimes h_L^k, \omega^b/b!)$. We denote by $\text{Met}(E)$ the set of smooth hermitian metrics on the bundle E .

Definition 6. We say that the metric $H \in \text{Met}(E)$ is *balanced* at level k if the Bergman endomorphism $B_k(H \otimes h_L^k, \omega^b/b!)$ is constant over the base, i.e.

$$B_k = \frac{h^0(E \otimes L^k)}{\text{rk}(E) \text{Vol}_L(B)} \text{Id}_E.$$

The connection with Gieseker stability is furnished by the following result of X. Wang [31]:

Theorem 7 (Wang). *The bundle E is Gieseker polystable if and only if there exists a sequence of metrics H_k on E such that H_k is balanced at level k for all $k \gg 0$.*

Taking E to be the trivial bundle, with the trivial metric, gives an important special case. Here the only metric that can vary is that on the line bundle L , and B_k becomes scalar valued. To emphasise the importance of this case we use separate terminology:

Definition 8. The *Bergman function* of a metric h_L on a line bundle L with curvature ω is the restriction to the diagonal of the kernel of the projection operator $P_k: C^\infty(B, L^k) \rightarrow H^0(B, L^k)$. This scalar valued smooth function shall be denoted by $\rho_k = \rho_k(h_L, \omega^b/b!)$.

We say the metric h_L is *balanced at level k* if ρ_k is a constant function, i.e.

$$\rho_k = \frac{h^0(L^k)}{\text{Vol}_L(B)}.$$

In this context, balanced metrics are related to stability of the base B as in the following result proved by Zhang [35], Luo [18], Paul [24] and Phong-Sturm [25]. We refer to the survey [13] for recent progress on the notion of asymptotic Chow stability.

Theorem 9. *There exists a hermitian metric h_L on L that is balanced at level k if and only if the embedding of X into projective space via the linear series determined by L^k is Chow polystable.*

Remark 10. It will turn out that the assumption that $\text{scal}(\omega)$ is constant is not strictly speaking necessary for our proof of Theorem 1. In fact, as will be apparent, a simple modification shows it is sufficient to assume that there is a sequence of metrics $h_{L,k}$ on L that are balanced at level k and whose associated curvatures ω_k are themselves bounded in the right topology. However we know of no examples of manifolds that admit such a sequence of metrics that do not admit a cscK metric, and thus this generalisation does not give anything new.

2.3 Density of States Expansion

Through work of S.T. Yau [33], G. Tian [30], D. Catlin [4], S. Zelditch [34], X. Wang [32] among others, one can understand the behaviour of the Bergman endomorphism as k tends to infinity through the so-called “density of states” asymptotic expansion. We refer to [19] as a reference for this topic. The upshot is that for fixed $q, r \geq 0$ one can write

$$B_k = k^b A_0 + k^{b-1} A_1 + \cdots + k^{b-q} A_q + O_{C^r}(k^{b-q-1}) \quad (4)$$

where $A_i \in C^\infty(\text{End}(E))$ are hermitian endomorphism valued functions. The A_i depend on the curvature of the metrics in question, and when necessary will be denoted by $A_i = A_i(h, H)$; in fact

$$A_0 = \text{Id}_E \quad \text{and} \quad A_1 = \sqrt{-1} \Lambda_\omega F_H + \frac{\text{scal}(\omega)}{2} \text{Id}_E.$$

Now a key point for our application is the observation that the above expansion still holds if the metrics on L and E are allowed to vary, so long as the *curvature* of the metric on E remains under control. This is made precise in the following proposition which is a slight generalisation of [19, Theorem 4.1.1].

Proposition 11. *Let $q, r \geq 0$ fixed as above. Let $h_{L,k} \in \text{Met}(L)$ be a sequence of metrics converging in C^∞ topology to $h_L \in \text{Met}(L)$ such that $\omega := c_1(h_L) > 0$. Let $H_k \in \text{Met}(E)$ be a sequence of metrics such that the curvatures F_{H_k} are bounded independently of k in $C^{r'}$ norm for some $r' \gg r, q$. Consider the Bergman endomorphism B_k associated to $H_k \otimes h_{L,k}^k \in \text{Met}(E \otimes L^k)$. Then B_k satisfies the uniform asymptotic expansion (4) with $A_i = A_i(h_{L,k}, H_k)$.*

We only provide a sketch of the proof of this proposition by pointing out how to adapt Tian's construction of peak sections ([30], [32, Section 5]) to this setting. It is important for our application that we do not assume the H_k necessarily converge.

In order to modify a smooth peaked section to a holomorphic one, and control the L^2 norm of this change, one needs to apply Hörmander L^2 estimates for the $\bar{\partial}$ operator. But under our assumptions $\sqrt{-1}\Lambda F_{H_k} + k\text{Id}_E$ is positive definite for k large enough so Hörmander's theorem (see [6, Theorem (8.4)]) holds on $E \otimes L^k$.

Since the calculation of the asymptotics is local in nature, another key ingredient is a pointwise expansion of the involved metrics. Fix a point $z_0 \in B$. From [7, Chapter V - Theorem 12.10], we know that there exists a holomorphic frame $(e_i)_{i=1,\dots,rk(E)}$ over a neighbourhood of $z_0 \in B$ such that, with respect to this frame, the endomorphism $\mathbf{H}_k(z)_{ij} = H_k(e_i, e_j)$ associated to the metric H_k has the following expansion:

$$\mathbf{H}_k(z)_{ij} = \left(\delta_{ij} - \sum_{1 \leq k, l \leq n} (F_{H_k})_{i\bar{j}k\bar{l}} z_k \bar{z}_l + \mathbf{O}(|z|^3) \right), \quad (5)$$

Furthermore, by induction one can show that the higher order terms of the expansions are given by derivatives of the curvature of the metric on E . For instance, at order 3, $\mathbf{H}_k(z)_{ij}$ has an extra term of the form

$$-\frac{1}{2} \left((F_{H_k})_{i\bar{j}a\bar{b},c} z_a \bar{z}_b z_c + (F_{H_k})_{i\bar{j}a\bar{b},\bar{c}} z_a \bar{z}_b \bar{z}_c, \right.$$

and thus in (5), $\mathbf{O}(|z|^3) = O_{C^{r'-1}}(k^0)$ under our assumptions. Similarly, the higher order terms of this Taylor expansion are under control. Using this, one can follow line by line the arguments of [32, Section 5] to obtain the proposition.

3 Construction of balanced metrics

In this section we construct metrics for $\mathbb{P}(E)$ that are almost balanced by perturbing the metrics on the bundle E . Then an application of the implicit function argument from [28] will provide the required balanced metrics.

3.1 Relating the metric on the bundle to the metric on the projectivisation

We first recall the techniques in [28] that relate the Bergman endomorphism on $E \otimes L^k$ and the Bergman function on the projectivisation $(\mathbb{P}(E), \mathcal{L}_k := \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* L^k)$.

Let V be a vector space equipped with a hermitian metric H_V . This induces in a natural way a Fubini-Study hermitian metric on $\mathcal{O}_{\mathbb{P}(V)}(1)$ which we denote by \hat{h}_V . Similarly given a hermitian metric H on E we get an induced metric \hat{h}_E on $\mathcal{O}_{\mathbb{P}(E)}(1)$. We denote by

$$\rho_k = \rho_k(\hat{h}_E \otimes \pi^* h_L^k)$$

the Bergman function on $(\mathbb{P}(E), \mathcal{L}_k)$ induced from the metric $\hat{h}_E \otimes \pi^* h_L^k$. The next results gives an asymptotic expansion for ρ_k in k (observe this is not the same as the usual density of states expansion, since we are not taking powers of a fixed line bundle).

Theorem 12 (Seyyedali [28]). *There exists smooth endomorphism valued functions $\tilde{B}_k = \tilde{B}_k(H, h_L)$ such that*

$$\rho_k([v]) = \frac{1}{c_r} \text{tr} \left(\frac{v \otimes v^{*H}}{\|v\|_H^2} \tilde{B}_k(H, h_L) \right) \quad \text{for } [v] \in \mathbb{P}(E), \quad (6)$$

where $c_r := \int_{\mathbb{C}^{r-1}} \frac{d\zeta \wedge d\bar{\zeta}}{(1 + \sum_{j=1}^{r-1} |\zeta_j|^2)^{r+1}}$. Moreover \tilde{B}_k has an asymptotic expansion of the form

$$\tilde{B}_k(H, h_L) = k^b \text{Id}_E + k^{b-1} \left(\sqrt{-1} [\Lambda_\omega F_H]^0 + \frac{\text{rk}(E) + 1}{2\text{rk}(E)} \text{scal}(\omega) \text{Id}_E \right) + \dots$$

where $[T]^0$ denotes the traceless part of the operator T .

We refer to \tilde{B}_k as the *distorted Bergman endomorphism*. The proof of the previous results is obtained by relating \tilde{B}_k to the Bergman endomorphism by an identity of the form

$$\left(\sum_{j=0}^b k^{j-b} \Psi_j \right) \tilde{B}_k(H, h_L) = B_k(H \otimes h_L^k, \omega^b/b!),$$

for certain $(\Psi_j)_{j=0..b} \in \text{End}(E)$ that depend only on the curvature of the metric $H \in \text{Met}(E)$. In fact,

$$\Psi_j = \Lambda_\omega^{b-j} \left(F_H^{b-j} + P_1(H) F_H^{b-j-1} + \dots + P_{b-j}(H) \right),$$

where $P_i(H) = P_i(C_1(H), \dots, C_{b-j}(H))$ are polynomials of degree i in the k -th Chern forms $C_k(H)$ of H , $1 \leq k \leq b-j$ [28, p 594]

Given this, the asymptotic expansion for \tilde{B}_k follows from that of B_k . Thus, using Proposition 11 we see that Theorem 12 in fact holds uniformly if the metric h is allowed to vary in a compact set, and the metric H is allowed to vary in such a way that the curvature F_H is bounded (as in the case for almost Hermitian-Einstein metrics).

3.2 Perturbation Argument

From now on let $E \rightarrow B$ be a vector bundle satisfying assumption (A), equipped with a family of almost Hermitian-Einstein metrics $H_k \in \text{Met}(E)$, and (L, h_L) a polarisation of the underlying manifold B such that $\omega = c_1(h_L)$ is a cscK metric. We now show how to adapt the methods of [28, Theorem 1.2], and prove the existence of metrics on \mathcal{L}_k that are almost balanced, in the sense that the associated Bergman function is constant up to terms that are negligible for large k [8].

The approach is to perturb both the Kähler metric ω and the almost Hermitian-Einstein metric H_k on E by considering

$$\begin{aligned}\omega'_k &= \omega + \sqrt{-1}\partial\bar{\partial}\left(\sum_{i=1}^q k^{-i}\phi_i\right), \\ H'_k &= H_k \left(\text{Id}_E + \sum_{i=1}^q k^{-i}\Phi_i \right),\end{aligned}$$

where ϕ_i are smooth functions on B and Φ_i are smooth endomorphisms of E . We will also denote the perturbed metric on L as $h'_L = h_L e^{-\sum_{i=1}^q k^{-i}\phi_i} \in \text{Met}(L)$ which satisfies $\omega'_k = c_1(h'_L)$. The perturbations terms Φ_i and ϕ_i will be constructed iteratively to make the distorted Bergman endomorphism approximately constant. In fact it will be necessary for ϕ_i and Φ_i to themselves depend on k , but for fixed i they will be of order $O_{C^r}(k^0)$, and this will be clear from their construction.

Observe the metrics ω'_k lie in a compact set, and the curvature of the metrics H'_k are bounded over k . For the first step of the iteration, where $q = 1$, we can apply Theorem 12 to deduce

$$\tilde{B}_k(H'_k, h'_L) = k^b \text{Id}_E + k^{b-1} A_1(H_k, \omega) + k^{b-2} (A_2(H_k, \omega) + \delta) + \cdots \quad (7)$$

where

$$A_1(H_k, \omega) = \sqrt{-1}[\Lambda_\omega F_{H_k}]^0 + \frac{\text{rk}(E) + 1}{2\text{rk}(E)} \text{scal}(\omega) \text{Id}_E$$

and we have defined

$$\delta = \delta(\Phi_1, \phi_1) := D(A_1)_{H_k, \omega}(\Phi_1, \phi_1),$$

where $D(A_1)$ is the linearisation of A_1 . Thus

$$\begin{aligned} D(A_1)_{H,\omega}(\Phi, \phi) &= \left. \frac{d}{dt} \right|_{t=0} A_1(H(\text{Id}_E + t\Phi), \omega + t\sqrt{-1}\partial\bar{\partial}\phi) \\ &= \frac{\text{rk}(E) + 1}{2\text{rk}(E)} (\mathbb{L}\phi) \text{Id}_E \\ &\quad + \sqrt{-1} [\Lambda_\omega \bar{\partial}\partial\Phi + \Lambda_\omega^2 (F_H \wedge \sqrt{-1}\partial\bar{\partial}\phi) - \Delta_\omega \phi \Lambda_\omega F_H]^0 \end{aligned}$$

where \mathbb{L} denotes the Lichnerowicz operator with respect to ω .

Now from the definition of the almost Hermitian-Einstein metrics we have an expansion (2)

$$T := \sqrt{-1}\Lambda_\omega F_{H_k} - \mu(E)\text{Id}_E = T_0 + T_1 k^{-1} + T_2 k^{-2} + \dots + T_{b-1} k^{b-1}, \quad (8)$$

where the $T_i = O_{C^r}(k^0)$ and T_0 is constant, see (3).

The metric H_k on E induces a metric on $\text{End}(E)$ (that we still denote by H_k in the sequel) and one has an operator $\partial_{\text{End}(E)}$ (that we still denote ∂ in the sequel) as the $(1, 0)$ part of the connection operator induced on $\text{End}(E)$ from the Chern connection on E compatible with H_k . One can write the associated curvature on $\text{End}(E)$ as $F_{\text{End}(E), H_k} = F_{H_k} \otimes \text{Id}_{E^*} + \text{Id}_E \otimes F_{E^*, H_k^*}$. Thus, we obtain a similar expansion as (8),

$$\begin{aligned} R &:= \sqrt{-1}\Lambda_\omega F_{\text{End}(E), H_k} - \mu(\text{End}(E))\text{Id}_{\text{End}(E)} \\ &= \sqrt{-1}\Lambda_\omega F_{\text{End}(E), H_k} \\ &= R_0 + R_1 k^{-1} + R_2 k^{-2} + \dots + R_{b-1} k^{b-1}, \end{aligned} \quad (9)$$

where the $R_i = O_{C^r}(k^0)$ and R_0 is constant.

Using this, we rewrite the distorted Bergman endomorphism as

$$\tilde{B}_k(H'_k, h'_L) = k^b \text{Id}_E + k^{b-1} \tilde{A}_1 + k^{b-2} \tilde{A}_2 + \dots$$

where

$$\begin{aligned} \tilde{A}_1 &= \frac{\text{rk}(E) + 1}{2\text{rk}(E)} \text{scal}(\omega) \text{Id}_E \\ \tilde{A}_2 &= [T_1]^0 + A_2 + \delta(\Phi_1, \phi_1) \end{aligned}$$

since $[T_0]^0 = 0$.

Observe since $\text{scal}(\omega)$ is constant, the top coefficient \tilde{A}_1 is also constant. The aim now is to find a perturbation that makes the lower order terms also constant. To this end it is convenient to define

$$\tilde{R} = R - R_0 = O_{C^r}(1/k)$$

and to rewrite the Bergman endomorphism once again, this time in the following way using (9)

$$\tilde{B}_k(H'_k, h'_L) = k^b \text{Id}_E + k^{b-1} \tilde{A}_1 + k^{b-2} (\tilde{A}_2 - [\tilde{R}\Phi_1]^0) + k^{b-3} (\tilde{A}_3 + [R_1\Phi_1]^0) + \dots \quad (10)$$

Remark that since Φ_1 is $O_{C^r}(k^0)$, the same will be true of $[R_1\Phi_1]^0$. The reason for adding and subtracting this term arises when it comes to ensuring that the Φ_i we construct are hermitian operators, as in the next proposition which ensures that it is possible to find ϕ_1 and Φ_1 to make the k^{b-2} term constant.

Define $\text{End}_0(E)$ to be the vector space of endomorphisms η of E such that $\int_B \text{tr} \eta \frac{\omega^b}{b!} = 0$, $\text{End}_0^0(E)$ the trace free elements of $\text{End}_0(E)$ and $C_0^\infty(B, \mathbb{R})$ the space of smooth functions with null integral with respect to the volume form $\frac{\omega^b}{b!}$.

Proposition 13. *Assume that $\text{Aut}(B, L)/\mathbb{C}^*$ is discrete, ω is a cscK metric and that E satisfies assumption (A). Then for any endomorphism $\zeta \in \text{End}_0(E)$ there exists a unique couple $(\Phi_1, \phi_1) \in \text{End}_0^0(E) \times C_0^\infty(B, \mathbb{R})$ such that*

$$\delta(\Phi_1, \phi_1) - [\tilde{R}\Phi_1]^0 = \zeta. \quad (11)$$

Furthermore, Φ_1 is hermitian with respect to H_k if and only if the same is true of ζ .

Finally if $r \geq 4$ and $\alpha \in (0, 1)$ there is a $c_{r,\alpha}$ such that for all ζ ,

$$\|\phi_1\|_{C^{r,\alpha}} + \|\Phi_1\|_{C^{r-2,\alpha}} \leq c_{r,\alpha} \|\zeta\|_{C^{r-4,\alpha}}.$$

We observe that \tilde{R} is non-zero for all k since by assumption E is not Mumford stable and thus none of the H_k are Hermitian-Einstein. The proof of the previous proposition will depend on a number of Lemmas, the first of which is a consequence of Kähler identities.

Lemma 14. *Let H be a hermitian metric on E which induces a metric on $\text{End}(E)$ that we still denote H . Then for any $\zeta \in \text{End}(E)$,*

$$\sqrt{-1}\Lambda_\omega \bar{\partial} \partial \zeta^{*H} = (\sqrt{-1}\Lambda_\omega \bar{\partial} \partial \zeta - [\sqrt{-1}\Lambda_\omega F_{\text{End}(E), H}, \zeta])^{*H}.$$

Lemma 15 (Poincaré type inequality). *Assume that E is a simple holomorphic vector bundle. Then there is a constant C such that if $H \in \text{Met}(E)$ and $\eta \in \text{End}(E)$, we have the following inequality with respect to the metric induced on $\text{End}(E)$,*

$$\|\eta\|_{L_H^2}^2 \leq C \|\bar{\partial} \eta\|_{L_H^2}^2 + \frac{1}{rk(E) \text{Vol}_L(B)} \left| \int_B \text{tr} \eta \frac{\omega^b}{b!} \right|^2$$

Note that if we consider another reference metric H_0 and H such that $r \cdot H_0 > H > r^{-1} \cdot H_0$ with $r > 1$, then we can choose C depending only on (H_0, r) .

Proof. This is standard from the fact $\bar{\partial}^* \bar{\partial}$ provides a positive elliptic operator and our simpleness assumption [32, Section 3]. Here the constant C in the statement can be taken as the first positive eigenvalue of the elliptic operator. Note that for a varying metric in a bounded family of $\text{Met}(E)$, since the $\bar{\partial}$ -operator doesn't depend on the metric, we can choose the constant C uniformly. \square

Lemma 16. *Assume that E is a simple holomorphic vector bundle. For k sufficiently large, given any $\zeta \in \text{End}_0(E)$ there is a unique $\eta \in \text{End}_0(E)$ such that*

$$\sqrt{-1}\Lambda_\omega \bar{\partial}\partial\eta - [\tilde{R}\eta]^0 = \zeta \quad (12)$$

Furthermore η is hermitian (with respect to H_k) if and only if the same is true for ζ . Finally if $r \geq 2$ and $\alpha \in (0, 1)$ there is a constant $c_{r,\alpha}$ such that

$$\|\eta\|_{C^{r,\alpha}} \leq c_{r,\alpha} \|\zeta\|_{C^{r-2,\alpha}}.$$

Proof. We use Fredholm alternative for elliptic equations. Firstly, \tilde{R} is hermitian thus, the operator

$$\eta \mapsto \sqrt{-1}\Lambda_\omega \bar{\partial}\partial\eta - [\tilde{R}\eta]^0$$

is hermitian and elliptic. To show existence of a solution, we need to show that this operator restricted on $\text{End}_0(E)$ has trivial kernel. Let us assume that

$$\sqrt{-1}\Lambda_\omega \bar{\partial}\partial\eta - [\tilde{R}\eta]^0 = 0. \quad (13)$$

Let us fix a smooth hermitian metric on E which gives us a metric on $\text{End}(E)$. Equation (13) implies, by Kähler identities and by taking inner product with η , that we have pointwise

$$\langle \partial\eta, \partial\eta \rangle - \langle [\tilde{R}\eta]^0, \eta \rangle = 0.$$

Now, this implies

$$\langle \bar{\partial}\eta^*, \bar{\partial}\eta^* \rangle - \langle [\tilde{R}\eta]^0, \eta \rangle = \|\bar{\partial}\eta^*\|^2 - \langle [\tilde{R}\eta]^0, \eta \rangle = 0. \quad (14)$$

Using Cauchy-Schwartz inequality, we have $\langle \tilde{R}\eta, \eta \rangle \leq \|\tilde{R}\| \|\eta\|^2 \leq \|\tilde{R}\| \|\eta^*\|^2$ and $\langle \text{Tr}(\tilde{R}\eta)\text{Id}, \eta \rangle \leq \text{rk}(E) \|\tilde{R}\| \|\eta\|^2$. By integration, we deduce, using that $\|\tilde{R}\|_{C^0} = O_{C^r}(1/k)$ and Lemma 15, that $\|\bar{\partial}\eta^*\|_{L^2}^2(1 - C/k) = 0$ from (14). Thus $\bar{\partial}\eta^* = 0$ if $k \gg 0$. But, since E is simple, this gives that $\eta = \alpha \text{Id}_E$ for a constant α , see [17, Section 7.2]. Finally, the kernel of the operator $\sqrt{-1}\Lambda_\omega \bar{\partial}\partial \cdot - [\tilde{R}\cdot]$ on $\text{End}_0(E)$ is trivial and we get uniqueness.

Let us show that we get a hermitian solution. From Lemma 14, one has that

$$\sqrt{-1}\Lambda_\omega \bar{\partial}\partial\eta^* = \left(\sqrt{-1}\Lambda_\omega \bar{\partial}\partial\eta - [\sqrt{-1}\Lambda_\omega F_{\text{End}(E), H_k}, \eta] \right)^*$$

where now the adjoint is computed with respect to the almost Hermitian-Einstein metric H_k on E . Since $[R_0, \eta] = 0$ (any term of the form $\theta \text{Id}_{\text{End}(E)}$ with θ a function is in the centre of the Lie algebra $\text{End}(E \otimes L^k)$), one can rewrite this equation as

$$\sqrt{-1}\Lambda_\omega \bar{\partial}\partial\eta^* = \left(\sqrt{-1}\Lambda_\omega \bar{\partial}\partial\eta - [\tilde{R}, \eta] \right)^*.$$

After expansion, this is equivalent to

$$(\sqrt{-1}\Lambda_\omega \bar{\partial}\partial - \tilde{R})\eta^* = \left((\sqrt{-1}\Lambda_\omega \bar{\partial}\partial - \tilde{R})\eta\right)^*$$

since \tilde{R} is hermitian, and this can be rewritten as

$$(\sqrt{-1}\Lambda_\omega \bar{\partial}\partial\eta^* - [\tilde{R}\eta^*]^0) = \left(\sqrt{-1}\Lambda_\omega \bar{\partial}\partial\eta - [\tilde{R}\eta]^0\right)^*.$$

Now, from the uniqueness we have shown previously, one gets that the solution is hermitian with respect to the metric H_k .

Let us denote $\text{End}_0(E)^{r,\alpha}$ the Sobolev space of $C^{r,\alpha}$ hermitian endomorphisms of $\text{End}_0(E)$. For $k \gg 0$, $r \geq 2$, we have that $\sqrt{-1}\Lambda_\omega \bar{\partial}\partial \cdot - [\tilde{R}\cdot]^0$ is an invertible linear differential operator of order 2 from $\text{End}_0(E)^{r,\alpha}$ to $\text{End}_0(E)^{r-2,\alpha}$ with uniformly bounded coefficients since we have the uniform control $\tilde{R} = O_{C^r}(1/k)$. The eigenvalues of $\sqrt{-1}\Lambda_\omega \bar{\partial}\partial \cdot$ are strictly positive, while the eigenvalues of the hermitian operator (of order 0) $[\tilde{R}\cdot]^0$ tend to 0 as k becomes larger. Thus this operator is uniformly elliptic, we can apply Schauder theory of elliptic regularity [17, Section 7.3]. Note that we could also invoke the work of Uhlenbeck and Yau for the operator $\sqrt{-1}\Lambda_\omega \bar{\partial}\partial \cdot$ with a slight generalisation. Finally, the inverse of this operator is bounded and we obtain the existence of a uniform constant $c > 0$ such that for any (η, ζ) satisfying (12),

$$\|\eta\|_{C^{r,\alpha}} \leq c\|\zeta\|_{C^{r-2,\alpha}}.$$

□

Proof of Proposition 13. Obviously, we have the decomposition $\text{End}_0(E) = \text{End}_0^0(E) \oplus C_0^\infty(B, \mathbb{R})\text{Id}_E$. First we deal with existence, by looking at the kernel of the operator on $\text{End}_0(E)$ given by

$$D(A_1)_{H_k, \omega}(\Phi_1, \phi_1) - [\tilde{R}\Phi_1]^0 = 0$$

where $\Phi_1 \in \text{End}_0^0(E)$ and $\phi_1 \in C_0^\infty(B, \mathbb{R})$. This is equivalent to ask that

$$\frac{\text{rk}(E)}{2\text{rk}(E) + 1} \mathbb{L}\phi_1 = 0 \quad (15)$$

$$\left[\sqrt{-1} (\Lambda_\omega \bar{\partial}\partial\Phi_1 + \Lambda_\omega^2 (F_{H_k} \wedge \sqrt{-1}\partial\bar{\partial}\phi_1) - \Delta_\omega \phi_1 \Lambda_\omega F_{H_k}) - \tilde{R}\Phi_1 \right]^0 = 0 \quad (16)$$

Equation (15) gives immediately that $\phi_1 = 0$ since the kernel of the Lichnerowicz operator consists of just the constant functions (see [8]) thanks to the fact that $\text{Aut}(B, L)/\mathbb{C}^*$ is discrete and since $\int_B \phi_1 \frac{\omega^b}{b!} = 0$. Now, since Φ_1 is trace free, Equation (16) reduces to

$$\sqrt{-1}\Lambda_\omega \bar{\partial}\partial\Phi_1 - [\tilde{R}\Phi_1]^0 = 0$$

which admits only the trivial solution, from Lemma 16 ($\tilde{R} \neq 0$ since the vector bundle E is not Mumford stable). Thus, by Fredholm alternative, we can solve Equation (11). Moreover, we know that the terms $\frac{\text{rk}(E)}{2\text{rk}(E)+1}\mathbb{L}\phi_1$ and $\sqrt{-1}\Lambda_\omega^2(F_{H_k} \wedge \sqrt{-1}\partial\bar{\partial}\phi) - \sqrt{-1}\Delta_\omega\phi_1\Lambda_\omega F_{H_k}$ are hermitian. Hence, for the solution Φ_1 of (11), if ζ is hermitian, one can rewrite this equation as

$$\sqrt{-1}\Lambda_\omega\bar{\partial}\partial\Phi_1 - [\tilde{R}\Phi_1]^0 = \zeta'$$

where ζ' is hermitian with respect to H_k . Then, applying Lemma 16, we get that Φ_1 is hermitian. Finally the regularity of the solution is a consequence of Lemma 16 and the fact that the Lichnerowicz operator is a strongly elliptic operator of order 4. \square

Returning now to the construction of the almost balanced metrics, using Proposition 13, we obtain (Φ_1, ϕ_1) such that the second term of (10) satisfies

$$\tilde{A}_2 - [\tilde{R}\Phi_1]^0 = C_2 \text{Id}_E$$

or equivalently

$$\delta(\Phi_1, \phi_1) - [\tilde{R}\Phi_1]^0 = -A_2 - [T_1]^0 + C_2 \text{Id}_E$$

where C_2 is a topological constant. Note that we have used here the obvious fact that $\int_B \text{tr}(C_2 - A_2)\omega^b = 0$.

For the next step of our iterative process, we perturb the metrics H_k and ω_k at the order $q = 2$ and try to find Φ_2, ϕ_2 such that the third term of (10) is constant. Now this third term can be written

$$A_3(H_k, \omega) + \delta(\Phi_2, \phi_2) - [\tilde{R}\Phi_2]^0 + [T_2]^0 + [R_1\Phi_1]^0 + b_{1,2}$$

with $b_{1,2}$ obtained from the deformation of A_2 , and thus depends only on the $(H_k, \Phi_1, \omega, \phi_1)$ computed at the previous step of the iteration. We then use the same trick as before, introducing the term $[\tilde{R}\Phi_2]^0$ in order to obtain a hermitian solution, and see that Φ_2, ϕ_2 need to satisfy

$$\delta(\Phi_2, \phi_2) - [\tilde{R}\Phi_2]^0 = C_3 \text{Id}_E - b_{1,2} - A_3(H_k, \omega) - [R_1\Phi_1]^0$$

where C_3 is a topological constant. Now solutions to this equation are guaranteed just as before using Proposition 13.

Repeating this iteration one sees that at each step one is led to solve the equation

$$\delta(\Phi_i, \phi_i) - [\tilde{R}\Phi_i]^0 = \zeta_i$$

where ζ_i is hermitian with respect to H_k and depends on the computations of the previous steps, i.e on the data $(H_k, \omega, \Phi_1, \dots, \Phi_{i-1}, \phi_1, \dots, \phi_{i-1})$ and $\int_B \text{tr}\zeta_i \omega^b = 0$. Clearly then the metric that we construct with this process is hermitian. Thus we have the following result:

Theorem 17. *Let E be a vector bundle that satisfies assumption (A) on the projective manifold B with $\dim_{\mathbb{C}} B = b$, (L, h_L) a polarisation on B with $\omega = c_1(h_L) > 0$. Assume that $\text{Aut}(B, L)/\text{mathbb{C}}^*$ is discrete and ω is a cscK metric. Consider an almost Hermitian-Einstein metric $H_k \in \text{Met}(E)$. Then any fixed integers $q, r > 0$, and $k \gg 0$, the metrics H_k and h_L can be deformed to new metrics $H'_k \in \text{Met}(E)$ and $h'_L \in \text{Met}(L)$ such that the distorted Bergman endomorphism $\tilde{B}_k(H'_k, h'_L)$ satisfies*

$$\tilde{B}_k(H'_k, h'_L) = k^b \text{Id}_E + \epsilon_k \in \text{End}(E)$$

where $\epsilon_k = O_{C^r}(k^{b-q})$.

Next consider \hat{h}' the metric induced on $\mathcal{O}_{\mathbb{P}(E)}(1)$ from $H'_k \in \text{Met}(E)$. Then using (6) gives the following corollary.

Corollary 18. *Under the same assumptions as in Theorem 17, for any fixed integers $q, r > 0$, and $k \gg 0$ each metric H_k and ω can be deformed to obtain a smooth hermitian metric $H'_k \in \text{Met}(E)$ and a smooth and $h'_L \in \text{Met}(L)$ such that the induced Bergman function $\rho_k(\hat{h}' \otimes \pi^* h'_L{}^k)$ on $\mathbb{P}(E)$ satisfies*

$$\rho_k(\hat{h}' \otimes \pi^* h'_L{}^k) = \hat{C} k^b + \hat{\epsilon}_k \in C^\infty(\mathbb{P}(E), \mathbb{R})$$

where \hat{C} is a topological constant and $\hat{\epsilon}_k = O_{C^r}(k^{b-q})$.

Proof of Theorem 1. The rest of the proof is the same as [28, Theorem 1.2] which shows how it is possible to perturb the almost balanced metrics above to obtain balanced metrics. Observe that all the estimates in sections 2,3 and 4 of [28] only require E to be simple, which is the case since we are assuming it to be Gieseker stable. Note also that $\mathbb{P}(E)$ has no nontrivial holomorphic vector fields ([28, Proposition 7.1]) since E is simple.

Finally, the fact that the existence of a balanced metric on $(\mathbb{P}(E), \mathcal{L}_k)$ implies the stability of the Chow point induced by $(\mathbb{P}(E), \mathcal{L}_k)$ [18, 35] since there is no nontrivial automorphism, completing the proof. \square

4 Computation of the Futaki invariant

We turn now to proving the instability result of Theorem 2. We refer the reader to [26] for an overview and of the concepts involved. What is required is to consider one parameter degenerations (so called “test configurations”) of our manifold $\mathbb{P}(E)$ and these can be constructed rather naturally from subbundles.

Suppose that F is a subbundle of E such that $G := E/F$ is locally free. This gives rise to a family of bundles $\mathcal{E} \rightarrow X \times \mathbb{C} \rightarrow \mathbb{C}$ with general fibre E and central fibre $F \oplus G$ over $0 \in \mathbb{C}$. Moreover \mathcal{E} admits a \mathbb{C}^* action that covers the usual action on the base \mathbb{C} , and whose restriction to $F \oplus G$ scales

the fibres of F with weight 1 and acts trivially on G . (One can see this in a number of ways, for instance if $\xi \in H^1(F \otimes G^*)$ represents the extension determined by E then this action takes ξ to zero as $\lambda \in \mathbb{C}^*$ tends to zero.)

Setting $\mathcal{X} = \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{C}$ and $\tilde{\mathcal{L}}_k = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^* L^k$ we thus have a flat family of polarised varieties with \mathbb{C}^* action whose general fibre is $(\mathbb{P}(E), \mathcal{L}_k)$ (i.e. a test-configuration as introduced in [9]).

The goal is to calculate the sign of a certain numerical invariant F_1 called the Futaki invariant (see [10]). We use the convention that if $F_1 < 0$ then $\mathbb{P}(E)$ is K-unstable, which is known to imply that it is asymptotically Chow unstable [27, Theorem 3.9].

To make the computations more palatable we restrict to the case that $\text{rank}(E) = 2$ over a smooth polarised base (B, L) of complex dimension $b \geq 2$, and assume that F and G are locally free (although the computation is essentially the same without this assumption, see [26, Section 5.4]). We denote by ch_2 the second Chern character, so $\text{ch}_2(F) = c_1(F)^2/2$ and $\text{ch}_2(E) = c_1(E)^2/2 - c_2(E)$.

We work initially over a base of complex dimension b since this adds no significant difficulties, although the reader may wish to set $b = 2$ which will be all that is necessary for our applications. To ease notation set $\omega = c_1(L)$ and if $\alpha_i \in H^{2d_i}(B)$ with $d_1 + \cdots + d_r = b$ we write $\alpha_1.\alpha_2 \cdots \alpha_r = \int_X \alpha_1 \wedge \cdots \wedge \alpha_r$.

Proposition 19. *The Futaki invariant of the test configuration $(\mathcal{X}, \tilde{\mathcal{L}}_k)$ is¹*

$$F_1 = C_1 k^{2b-1} + C_2 k^{2b-2} + O(k^{2b-3}) \quad (17)$$

where

$$\begin{aligned} C_1 &= \frac{\omega^b}{6b!(b-1)!} (\mu(E) - \mu(F)), \\ C_2 &= \frac{\omega^b}{12b!(b-2)!} (c_1(E)/2 - c_1(F))c_1(B).\omega^{b-2} \\ &\quad + \frac{\omega^b}{3b!(b-2)!} (\text{ch}_2(E)/2 - \text{ch}_2(F)).\omega^{b-2} \\ &\quad + \frac{1}{12(b-1)!^2} \left(2c_1(E).\omega^{b-1} - c_1(B).\omega^{b-1} \right) (\mu(E) - \mu(F)). \end{aligned}$$

Proof of Proposition 19. Recall $\pi_* \mathcal{L}_k^r = S^r E \otimes L^{rk}$ for $r \geq 0$, so from the

¹This corrects an error in the lower order term of [26, Prop. 5.23]

Riemann-Roch theorem, we get

$$\begin{aligned}
\chi(\mathcal{L}_k^r) &= \chi(S^r E \otimes L^{kr}) = \int_B e^{rk\omega} \text{ch}(S^r E) Td(B), \\
&= \frac{r^b k^b \omega^b}{b!} \text{rank}(S^r E) \\
&\quad + \frac{r^{b-1} k^{b-1}}{(b-1)!} \omega^{b-1} \left(\text{rank}(S^r E) \frac{c_1(B)}{2} + c_1(S^r E) \right) \\
&\quad + \frac{r^{b-2} k^{b-2}}{(b-2)!} \omega^{b-2} \left(\text{rank}(S^r E) \text{Todd}_B^{(2)} + \frac{c_1(S^r E) \cdot c_1(B)}{2} + \text{ch}_2(S^r E) \right) \\
&\quad + O(k^{b-3}),
\end{aligned}$$

where $\text{Todd}_B^{(2)}$ denotes the second Todd class of B , and we use the convention that $O(k^{b-3})$ vanishes if $b = 2$. Now, using the splitting principle, it is elementary to check that

$$\begin{aligned}
\text{rank}(S^r E) &= r + 1, \\
c_1(S^r E) &= r(r+1)c_1(E)/2, \\
\text{ch}_2(S^r E) &= r^3[c_1(E)^2/12 + \text{ch}_2(E)/6] + r^2 \text{ch}_2(E)/2 + O(r).
\end{aligned}$$

Thus for $r \gg 0$,

$$p(r) := h^0(\mathbb{P}(E), \mathcal{L}_k^r) = a_0 r^{b+1} + a_1 r^b + O(r^{b-1}),$$

where

$$\begin{aligned}
a_0 &= \frac{k^b \omega^b}{b!} + \frac{k^{b-1} \omega^{b-1} c_1(E)}{2(b-1)!} + \frac{k^{b-2} \omega^{b-2}}{(b-2)!} \left(\frac{1}{12} c_1(E)^2 + \frac{1}{6} \text{ch}_2(E) \right) \\
&= +O(k^{b-3}), \\
a_1 &= \frac{k^b \omega^b}{b!} + \frac{k^{b-1} \omega^{b-1}}{2(b-1)!} \cdot (c_1(B) + c_1(E)) \\
&\quad + \frac{k^{b-2} \omega^{b-2}}{(b-2)!} \left(\frac{\text{ch}_2(E)}{2} + \frac{c_1(E) \cdot c_1(B)}{4} \right) + O(k^{b-3}).
\end{aligned}$$

Turning to the central fibre $\mathbb{P}(F \oplus G)$, we have a splitting

$$\begin{aligned}
H^0(\mathbb{P}(F \oplus G), \tilde{\mathcal{L}}_k^r) &= H^0(B, S^r(F \oplus G) \otimes L^{kr}) \\
&= \bigoplus_{i=0}^r H^0(B, F^i \otimes G^{r-i} \otimes L^{kr}),
\end{aligned}$$

Moreover this is the eigenspace decomposition for the action, with the i -th space having weight i . Let $w(r)$ be the sum of the eigenvalues of the action on this vector space, so

$$w(r) = \sum_{i=0}^r i h^0(B, F^i \otimes G^{r-i} \otimes L^{kr}).$$

Now since $\tilde{\mathcal{L}}_k$ is relatively ample, the higher cohomology groups vanish, and thus pushing forward to B we have that the higher cohomology groups of $F^i \otimes G^{r-i} \otimes L^{kr}$ vanish for $r \gg 0$. Thus from Riemann-Roch again, $h^0(F^i \otimes G^{r-i} \otimes L^{kr})$ equals

$$\begin{aligned} & \frac{k^b r^b \omega^b}{b!} + \frac{k^{b-1} r^{b-1} \omega^{b-1}}{(b-1)!} \left(\frac{c_1(B)}{2} + ic_1(F) + (r-i)c_1(G) \right) \\ & + \frac{k^{b-2} r^{b-2} \omega^{b-2}}{(b-2)!} \left(\frac{(ic_1(F) + (r-i)c_1(G))^2}{2} + Td_B^{(2)} \right) \\ & + \frac{k^{b-2} r^{b-2} \omega^{b-2}}{(b-2)!} \left(\frac{c_1(B)(ic_1(F) + (r-i)c_1(G))}{2} \right) + O(r^{b-3}). \end{aligned}$$

Now an elementary calculation gives $w(k) = b_0 r^{b+2} + b_1 r^{b+1} + O(r^b)$, where

$$\begin{aligned} b_0 &= \frac{k^b \omega^b}{2b!} + \frac{k^{b-1} \omega^{b-1} c_1(F)}{3(b-1)!} + \frac{k^{b-1} \omega^{b-1} c_1(G)}{6(b-1)!} \\ & + \frac{k^{b-2} \omega^{b-2}}{2(b-2)!} \left(\frac{c_1(F)^2}{4} + \frac{c_1(F)c_1(G)}{6} + \frac{c_1(G)^2}{12} \right) + O(k^{b-3}), \\ b_1 &= \frac{k^b \omega^b}{2b!} + \frac{k^{b-1} \omega^{b-1} c_1(F)}{2(b-1)!} + \frac{k^{b-1} \omega^{b-1} c_1(B)}{4(b-1)!} \\ & + \frac{k^{b-2} \omega^{b-2} c_1(B)}{2(b-2)!} \left(\frac{c_1(F)}{3} + \frac{c_1(G)}{6} \right) + \frac{k^{b-2} \omega^{b-2} c_1(F)^2}{4(b-2)!} + O(k^{b-3}). \end{aligned}$$

The definition of the Futaki invariant is $F_1 = b_0 a_1 - b_1 a_0$, and putting this all together gives the result as stated. \square

Proposition 20. *Suppose B is a surface, and $\chi(F \otimes L^k) = \chi(E \otimes L^k)/2$ for all k . Suppose also that either $c_1(B) = 0$ or $\omega = \pm c_1(B)$. Then $(\mathbb{P}(E), \mathcal{L}_k)$ is not K -polystable for k sufficiently large.*

Proof. The previous computations can be extended to show that for a surface one can write the Futaki invariant as $F_1 = C_1 k^3 + C_2 k^2 + C_3 k + C_4$ with C_1, C_2 given by Proposition 19 that vanish and

$$\begin{aligned} 48C_3 &= (8 \deg_L E - 4c_1(L)c_1(B)) (\text{ch}_2(E)/2 - \text{ch}_2(F)) \\ & + 2c_1(E)^2 (\deg_L(E)/2 - \deg_L F) \\ & + 2 \deg_L(F) c_1(E) c_1(B) - 2 \deg_L(E) c_1(B) c_1(F) \\ 144C_4 &= c_1(E)^2 [(c_1(E)/2 - c_1(F)) \cdot c_1(B) + 6(\text{ch}_2(E)/2 - \text{ch}_2(F))] \\ & - 4c_1(E) \cdot c_1(B) (\text{ch}_2(E)/2 - \text{ch}_2(F)) \\ & + 2(c_1(E) \cdot c_1(B) \text{ch}_2(F) - c_1(F) \cdot c_1(B) \text{ch}_2(E)) \end{aligned} \quad (18)$$

It is now an easy computation to check that under our assumptions, the terms C_3 and C_4 vanish. Also observe that the degeneration used above

is not a product test configuration, so $(\mathbb{P}(E), \mathcal{L}_k)$ is not K-polystable for k large enough. \square

Proof of Theorem 2. Suppose that E is a rank 2 vector bundle that is Gieseker stable but not Mumford stable and $\mu(F) = \mu(E)$. From Proposition 19, the term C_1 vanishes and the Futaki invariant of the test configuration associated to F is

$$F_1 = \frac{k^2}{24} (4 (\text{ch}_2(E)/2 - \text{ch}_2(F)^2) + c_1(B) (c_1(E)/2 - c_1(F))) + O(k).$$

Thus by hypothesis, $F_1 < 0$ for $k \gg 0$, proving that $(\mathbb{P}(E), \mathcal{L}_k)$ is not K-semistable for $k \gg 0$ as claimed. \square

5 Examples

We end by constructing examples of polarised surfaces (B, L) and vector bundles E over B that satisfy the assumptions of Theorems 1 and 2. To do so we start with a base B with trivial automorphism group that has an abundance of cscK metrics.

Fix a rank 2 Mumford stable bundle V over a complex projective curve C of genus $g \geq 2$ and define $B = \mathbb{P}(V)$. As is well known, using the Narasimhan-Seshadri Theorem [23] one can prove there exists a cscK metric in each Kähler class of B (see [11, 1.6] for the argument). Moreover as V is simple and $g \geq 2$, there are no infinitesimal automorphisms of B [28, Proposition 7.1].

We seek a suitable vector bundle E over B which is Gieseker stable and not Mumford stable, and from a simple consideration of the dimension of the relevant moduli spaces it is apparent that such bundle exist. In fact the dimension of the (smooth) moduli space of rank 2 Gieseker stable bundle with fixed Chern class c_1, c_2 is (when non empty) $4c_2 - c_1 + 4g - 3$ [12, Corollary 18] and using $g \geq 2$ one see this is strictly larger than the dimension of the (smooth) moduli space of Mumford stable bundle of same type, which is $4c_2 - c_1 + 3g - 3$ [20, Proposition 6.9].

Next we fix some notations and describe the ample cone of B . The Néron-Severi group of B can be identified with $\mathbb{Z} \times \mathbb{Z}$, with generators the class \mathfrak{b} of $\mathcal{O}_B(1)$ and the class \mathfrak{f} of a fibre over C . We have $\mathfrak{b}^2 = \deg(V)$, $\mathfrak{f}^2 = 0$ and $\mathfrak{b} \cdot \mathfrak{f} = 1$ while the anti-canonical divisor is given by $-K_B = 2\mathfrak{b} + 2(1 - g)\mathfrak{f}$. To ease the computations we may as well take $\deg V = 0$. Then, from [29, Proposition 3.1], or [12, Proposition 15], we know that a class $x\mathfrak{b} + y\mathfrak{f}$ is ample if $x > 0$ and $y > 0$.

Following the ideas of [29, Proposition 3.9], consider a rank 2 vector bundle E_1 obtained as an extension

$$0 \rightarrow \mathcal{O}_B \rightarrow E_1 \rightarrow F_1 \rightarrow 0,$$

where F_1 has class $-\mathfrak{b} + (m+1)\mathfrak{f}$ for some large positive m . To ensure that we can take such an extension that does not split, we need that $\text{Ext}^1(\mathcal{O}_B, F_1^*) = H^1(F_1^*)$ is non trivial. But this follows easily from Riemann-Roch since $\chi(F_1^*) = h^0(F_1^*) - h^1(F_1^*) + h^2(F_1^*) \geq -h^1(F_1^*)$ and

$$\begin{aligned}\chi(F_1^*) &= c_1(F_1^*)^2 + \frac{c_1(B)}{2}c_1(F_1^*) + \text{Todd}_2(B), \\ &= -2(m+1) + (-(m+1) + (1-g)) + (1-g), \\ &= -3(m+1) + 2(1-g) < 0.\end{aligned}$$

Over B , we take the polarisation $L_{m+1} = \mathfrak{b} + (m+1)\mathfrak{f}$ and one checks easily that

$$\mu(F_1) = \mu(E_1) = 0.$$

We claim that E_1 is in fact Mumford semi-stable. A priori, we need to check stability with respect to any rank 1 torsion free subsheaf \mathcal{F} of E_1 but since we are working with a rank 2 bundle on a surface, \mathcal{F}^{**} is a reflexive rank 1 sheaf on B and thus a line bundle. So $\mathcal{F} = \mathcal{O}(D) \otimes \mathcal{I}$ where $\mathcal{O}(D)$ is a line bundle and \mathcal{I} is an ideal sheaf with 0-dimensional support, so $c_1(\mathcal{F}) = c_1(\mathcal{F}^{**}) = c_1(\mathcal{O}(D))$. Since $E_1 = E_1^{**}$, it is now clear that it is sufficient to consider stability with respect to subbundles of E . But, for any rank 1 subbundle $\mathcal{O}(D)$ of E_1 , either $\mathcal{O}(D) \hookrightarrow \mathcal{O}$ or $F_1 \otimes \mathcal{O}(-D)$ is effective. In the first case it is immediate that $\mathcal{O}(D)$ does not destabilise E_1 . In the second case if we write the first Chern class of $\mathcal{O}(D)$ as $x_D\mathfrak{b} + y_D\mathfrak{f}$ we see by intersecting with ample line bundles that $x_D \leq -1$ and $y_D \leq m+1$. Hence $\mu(\mathcal{O}(D)) \leq \mu(F_1) = \mu(E_1)$ and E_1 is Mumford semi-stable with respect to L_{m+1} as claimed.

In order to construct a Gieseker stable bundle which is not Mumford stable, we tensor the previous extension by a line bundle F_2 with first Chern class $c_1(F_2) = -\mathfrak{b} + (g-3-m)\mathfrak{f}$, resulting in a non-trivial extension

$$0 \rightarrow F_2 \rightarrow E \rightarrow F_1 \otimes F_2 \rightarrow 0.$$

Observe that $\mu(F_2) = \mu(E)$ and so $\{0\} \subset F_2 \subset E$ is the Jordan-Hölder filtration of the Mumford semistable bundle E .

We claim that E is in fact Gieseker stable. As before, let $\mathcal{F} = \mathcal{O}(D) \otimes \mathcal{I}$ is a rank 1 torsion free subsheaf of E , and taking the double dual $\mathcal{O}(D)$ is a subbundle of E . Then either $\mathcal{O}(D) \hookrightarrow F_2$ or $F_1 \otimes F_2 \otimes \mathcal{O}(-D)$ is effective. In the first case by writing $c_1(\mathcal{O}(D)) = x_D\mathfrak{b} + y_D\mathfrak{f}$ one checks that if $(x_D, y_D) \neq (-1, g-3-m)$ then $\mu(\mathcal{O}(D)) < \mu(F_2) = \mu(E)$ while if $(x_D, y_D) = (-1, g-3-m)$ then $\mu(\mathcal{O}(D)) = \mu(E)$ and

$$\frac{\text{ch}_2(E)}{2} - \text{ch}_2(\mathcal{O}(D)) + \frac{c_1(B)}{2} \left(\frac{c_1(E)}{2} - c_1(\mathcal{O}(D)) \right) = \frac{1}{2} > 0.$$

Thus by Riemann-Roch we conclude $\frac{1}{2}\chi(E \otimes L_{m+1}^p) > \chi(\mathcal{O}(D) \otimes L_{m+1}^p)$ for $p \gg 0$ and so $\mathcal{O}(D)$ does not Gieseker destabilise. Moreover this inequality

only improves if $\mathcal{O}(D)$ is replaced by \mathcal{F} since $c_2(\mathcal{F})$ is the length of the support of \mathcal{I} and thus is non-negative. In the second case, in which $F_1 \otimes F_2 \otimes \mathcal{O}(-D)$ is effective, one deduces $x_D \leq -2$ and $y_D \leq g - 2$ with at least one inequality being strict, and so $\mu(\mathcal{O}(D)) < \mu(F_1 \otimes F_2) = \mu(E)$. Hence E is Gieseker stable with respect to L_{m+1} as claimed.

So Theorem 1 can be applied in this setting and $(\mathbb{P}(E), \mathcal{L}_k)$ is Chow stable for k sufficiently large where $\mathcal{L}_k = \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* L_{m+1}^k$. To apply Theorem 2, we compute

$$4(\text{ch}_2(E)/2 - \text{ch}_2(F_2)) + c_1(B) \cdot (c_1(E)/2 - c_1(F_2)) = -m - g + 2 < 0.$$

Hence $(\mathbb{P}(E), \mathcal{L}_k)$ is not K-semistable, thereby proving:

Corollary 21. *There exists smooth polarised manifolds (X, L) such that (X, L) is Chow stable but not asymptotically Chow stable.*

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